A MATHEMATICAL MODEL FOR PULSATILE BLOOD FLOW THROUGH A NARROW CATHETERIZED ARTERY WITH NONSYMMETRICAL STENOSIS

N.R.Neelavathi*   R.Jayalakshmi**

**Assistant Professor In Mathematics, P.K.R Arts College For Women, Gobichettipalayam, Erode District, Tamilnadu.*
***Associate Professor And Head, Department Of Mathematics, P.K.R Arts College For Women, Gobichettipalayam, Erode District, Tamilnadu.*

Abstract

The modeling of pulsatile blood flow through a narrow catheterized artery in presence of nonsymmetrical mild stenosis with a velocity slip at stenotic wall has been investigated in this paper. The blood flow is considered to be incompressible, Newtonian with variable blood viscosity. By using perturbation analysis, analytic expressions for the velocity profile, flow rate, wall shear stress and effective viscosity, are derived. The influence of stenosis height, shape, slip velocity and radius of catheter on axial velocity, wall shear stress, effective viscosity, flow rate and arterial wall shear stress in particular, in the narrow catheterized artery are represented graphically and discussed.

Keywords: Stenosis, Newtonian fluid, Catheterized artery, Shear stress, Longitudinal impedence.

1. Introduction

Circulatory disorders are considered to be for more than 55% of all deaths occurred in developed countries. An abnormal growth, formed due to deposits of atherosclerotic plaques in the lumen of an artery is normally called stenosis. Its growth may include the disorders in circulatory systems may be included as, narrowing in body passage leading to the reduction and impediment to blood flow in the constricted artery regions, the blockage of the artery in making the flow irregular and causing an abnormality of the blood flow and, the presence of stenosis at one or more of the major blood vessels, carrying blood to the heart or brain etc., could lead to various arterial diseases e.g., myocardial infarction, angina pectoris, cerebral accident, coronary thrombosis, strokes etc.[3, 12, 7].

Here, Catheterization refers to a procedure in which a long, thin, flexible plastic tube (catheter) is inserted into an artery [1]. A catheter with a very tiny balloon at the end is inserted into the artery in balloon angioplasty to treat stenosis. The catheter is carefully guided to the location at which stenosis occurs and balloon is inflated to fracture the fatty deposits and widen the narrowed portion of the artery [13]. The insertion of a catheter in an artery will naturally form an annular region between the walls of the catheter and artery. As a result, this will change the flow field, like modifying the pressure distribution and increasing the resistance. Thus it is of essential importance to study the flow of blood in a narrow catheterized artery.

The use of catheters is of vital importance and has become a standard tool for diagnosis and treatment in modern medicine. Transducers attached to catheters are of use in clinical works and the technique is used for measuring blood pressure and other mechanical properties in arteries. A catheter is composed of polyester based thermoplastic polyurethane, medical grade polyvinyl chloride, etc. When a catheter is inserted into the stenosed artery, the further increased impedance or frictional resistance to flow will alter the velocity distribution. Many researchers have investigated the flow of blood in an artery, in presence of a catheter by modeling the catheter and artery as rigid co-axial cylinders and blood as either a Newtonian or a non-Newtonian fluid. McDonald [2] addressed the pulsatile blood flow in a catheterized artery and obtained theoretical estimates for pressure gradient corrections for catheters. Karahalios [9] has discussed the effect of catheterization on various flow characteristics in an artery with or without stenosis. Jayaraman and Dash [8] worked a numerical study of blood flow in catheterized curved artery with constriction. Dash et al. [11] analysed the steady and pulsatile flow of blood in a narrow catheterized artery estimated the increase in frictional resistance in the artery due to catheterization, using a Casson fluid model. Sankar and Hemalatha [4] considered the steady flow of Herschel–Bulkley fluid through a catheterized artery. Sankar [5] has analysed a two-fluid model for the pulsatile flow of blood in a catheterized artery, by considering the core layer as a Casson fluid and the peripheral layer as a Newtonian fluid.

Although, blood acts as a non-Newtonian character at low shear rates [6], at high shear rates, generally found in larger arteries (diameter nearly above 1mm), blood behaves like a Newtonian fluid [10]. Since, stenosis normally generates and increases in large diameter arteries (in the range of 500 to 2000 μm), where blood shows a Newtonian behaviour, it appears to be reasonable in assuming blood to be homogeneous, isotropic, incompressible, Newtonian continuum, having a constant viscosity and density for flow though stenosed arteries (having respective radii 10.0, 5.0, 4.0 and 1.5mm in aorta, femoral, carotid and coronary arteries [14]).

To treat stenosis in balloon angioplasty, a catheter with a tiny balloon attached at the end is inserted into the narrow catheterized artery.The catheter is guided carefully to the location at which stenosis occurs and the balloon is then inflated to fracture the fatty deposits and widen the narrowed portion of the artery.

2. Formulation of A Mathematical Model

Consider an axially symmetric, laminar, pulsatile and fully improved flow of blood (assumed to be incompressible) through a catheterized circular tube with an axially asymmetric but radially symmetric mild stenosis as shown in Fig. 1. It is assumed that wall of the tube is rigid and the fluid blood is represented by a Newtonian fluid.

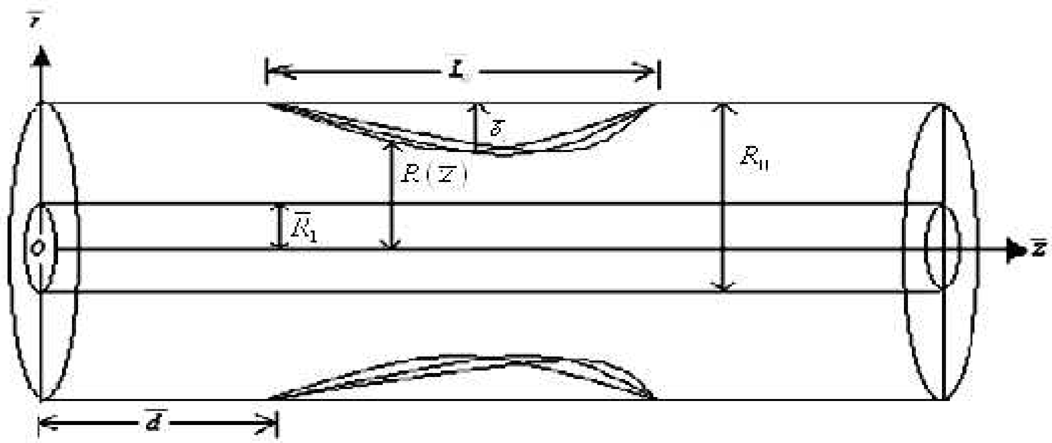


Fig.1 Flow Geometry of an axially nonsymmetrical stenosis with an inserted catheter

The geometry of the stenosis is given by

$$\bar{R}(\bar{z}) = \begin{cases} \bar{R}_0 - \bar{A}\left[\bar{L}_0^{n-1}(\bar{z}-\bar{d}) - (\bar{z}-\bar{d})^n\right]; & \bar{d} \le \bar{z} \le \bar{d} + \bar{L}_0 \\ \bar{R}_0, otherwise, \end{cases} \quad (1)$$

where $\bar{R}(\bar{z})$ is the radius of the artery in the stenosed region, $\bar{R}_0$ is the radius of the normal artery, n(≥ 2) is a parameter (called shape parameter) determining the stenosis shape (the symmetric stenosis occurs when n = 2), L_0 is stenosis length and d indicates its location. The parameter A is given by

$$\bar{A} = \frac{\bar{\delta}_s n^{n/(n-1)}}{\bar{L}_0^n (n-1)},$$

where $\bar{\delta}_s$ denotes the maximum height of the stenosis located at $z = d + L_0/n^{n/(n-1)}$, such that $\bar{\delta}_s \neq \bar{R}_0 << 1$. It has been reported that the radial velocity is negligibly small and can be neglected for a low Reynolds number flow in a tube with mild stenosis [7,15]. The equations of motion governing the fluid flow are given by

$$\bar{\rho}\frac{\partial \bar{u}}{\partial \bar{t}} = -\frac{\partial \bar{p}}{\partial \bar{z}} - \frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}(\bar{r}\bar{\tau}), \qquad (2)$$

$$\frac{\partial \bar{p}}{\partial \bar{r}} = 0. \qquad (3)$$

where $\bar{u}$ represents the fluid velocity in the axial direction, $\bar{\rho}$ is the density and $\bar{p}$ denotes the pressure.

The constitutive equation of Newtonian fluid is given by

$$\bar{\tau} = -\bar{\mu}\frac{\partial \bar{u}}{\partial \bar{r}} \qquad (4)$$

where $\bar{\mu}$ is the coefficient of viscocity and ι is the shear stress.

The boundary conditions are given by

$$\bar{u} = \bar{u}_s \text{ at } \bar{r} = \bar{R}(\bar{z}) \qquad (5)$$

$$\bar{u} = 0 \text{ at } \bar{r} = \bar{R}_1 \qquad (6)$$

where $\bar{u}_s$ is the slip velocity at the stenotic wall [16] and $R_1(<<R_0)$ is the radius of the catheter.

Since, the pressure gradient is a function of z and t , we take

$$-\frac{\partial \bar{p}}{\partial \bar{z}}(\bar{z},\bar{t}) = q(\bar{z})f(\bar{t}), \qquad (7)$$

where $\bar{q}(\bar{z}) = -\frac{\partial \bar{p}}{\partial \bar{z}}(\bar{z},0), f(\bar{t}) = 1 + a\sin\bar{\omega}\bar{t}$, a is the amplitude and $\bar{\omega}$ is the angular frequency of blood flow [17].

Let us introduce the following non-dimensional variables

$$z=\frac{\bar{z}}{\bar{R}_0},R(z)=\frac{\bar{R}(\bar{z})}{\bar{R}_0},\ R_1=\frac{\bar{R}_1}{\bar{R}_0},r=\frac{\bar{r}}{\bar{R}_0},t=\bar{t}\bar{\omega},\ L_0=\frac{\bar{L}_0}{\bar{R}_0},d=\frac{\bar{d}}{\bar{R}_0},$$

$$\delta_s=\frac{\bar{\delta}_s}{\bar{R}_0},A=\bar{A}\bar{R}_0^{n-1},u=\frac{\bar{u}}{\bar{q}_0\bar{R}_0^2/4\mu},u_s=\frac{\bar{u}_s}{\bar{q}_0\bar{R}_0^2/4\mu},\alpha^2=\frac{\bar{R}_0^2\bar{\omega}\bar{\rho}}{\bar{\mu}} \tag{8}$$

where α is the pulsatile Reynolds numbers for Newtonian fluid and $\bar{q}_0$ is the negative of the constant pressure gradient in a uniform tube without catheter.
The non-dimensional form of geometry of stenosis is given by

$$R(z)=\begin{cases}1-A[L_0^{n-1}(z-d)-(z-d)^n]; & d\le z\le d+L_0\\ 1, otherwise,\end{cases} \tag{9}$$

Using non-dimensional variables equations (2) and (4) reduce to

$$\alpha^2\frac{\partial u}{\partial t}=4q(z)f(t)-\frac{2}{r}\frac{\partial}{\partial r}(r\tau), \tag{10}$$

$$\tau=-\frac{1}{2}\frac{\partial u}{\partial r}. \tag{11}$$

With the help of (11), equation (10) becomes

$$\alpha^2\frac{\partial u}{\partial t}=4q(z)f(t)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \tag{12}$$

The boundary conditions in the non-dimensional form are given by

$$u=u_s \quad at \quad r=R(z), \tag{13}$$

$$u=0 \quad at \quad r=R_1. \tag{14}$$

The non-dimensional volumetric flow rate is given by

$$Q=4\int_{R_1}^{R(z)}r u(r,z,t)dr, \tag{15}$$

where $Q(t)=\frac{Q(\bar{t})}{\frac{\pi(\bar{R}_0)^4\bar{q}_0}{8\mu}};\bar{Q}(\bar{t})=2\pi\int_{\bar{R}_1}^{\bar{R}(\bar{z})}\bar{r}\bar{u}(\bar{r},\bar{z},\bar{t})d\bar{r}$ is the volumetric flow rate.

The effective viscosity μ_e defined as

$$\bar{\mu}_e=\frac{\pi\left(-\frac{\partial\bar{p}}{\partial\bar{z}}\right)(\bar{R}(\bar{z}))^4}{Q(\bar{t})}, \tag{16}$$

can be expressed in dimensionless form as

$$\mu_e = \frac{(R(z))^4}{Q(t)} q(z) f(t), \tag{17}$$

where $Q(t)$ is defined in equation (15).

3. Solution

Considering the Womersley parameter to be small, the velocity u can be expressed in the following form

$$u(z,r,t) = u_0(z,r,t) + \alpha^2 u_1(z,r,t) \ldots \ldots\ldots \tag{18}$$

Substituting the expression of u from equation (18) in (12), we have

$$\frac{\partial}{\partial r}\left(r\frac{\partial u_0}{\partial r}\right) = -4rq(z)f(t), \tag{19}$$

$$\frac{\partial u_0}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_1}{\partial r}\right). \tag{20}$$

Substituting u from equation (18) into conditions (13) and (14) we get

$$u_0 = u_s, u_1 = 0 \quad at \quad r = R(z) \; and \; u_0 = 0, u_1 = 0, r = R_1 \tag{21}$$

To determine u_0 and u_1 , we integrate equations (19) and (20) twice with respect to r and use the boundary conditions (21) (for u_1 , using the expression obtained for u_0), we have

$$u_0 = \left[1 - \frac{\log\left(\frac{r}{R}\right)}{\log\left(\frac{R_1}{R}\right)}\right]u_s + q(z)f(t)\left[(R^2 - r^2) - \frac{(R^2 - R_1^2)}{\log\left(\frac{R_1}{R}\right)}\log\left(\frac{r}{R}\right)\right], \tag{22}$$

$$u_1 = \frac{q(z)f'(t)}{16}\left[(4R^2r^2 - r^4 - 3R^4) - \frac{(R^2-R_1^2)}{\log\left(\frac{R_1}{R}\right)}\left\{4r^2\log\left(\frac{r}{R}\right) - 3r^2 + 3R^2\right\} - \frac{\log\left(\frac{r}{R}\right)}{\log\left(\frac{R_1}{R}\right)}\left\{4R^2R_1^2 - R_1^2 - 3R^4 - \frac{(R^2-R_1^2)}{\log\left(\frac{R_1}{R}\right)}\left(4R_1^2\log\left(\frac{R_1}{R}\right) + 3R^2 - 3R_1^2\right)\right\}\right]$$

(23)

where $R = R(z)$

The expression for velocity u can easily be obtained from equations (18), (22) and (23).

The wall shear stress τ_w (as a result of equations (11) and (18)) becomes

AIJRRR
ISSN :2456-205X

$$\tau_w = -\frac{1}{2}\left(\frac{\partial u_0}{\partial r} + \alpha^2 \frac{\partial u_1}{\partial r}\right)_{r=R(z)} \qquad (24)$$

which is determined, by substituting velocity expressions (22) and (23) into the above equation (24), in the form

$$\tau_w = \frac{u_s}{2R\log\left(\frac{R_1}{R}\right)} + q(z)f(t)\left\{R + \frac{(R^2-R_1^2)}{2R\log\left(\frac{R_1}{R}\right)}\right\} - \frac{\alpha^2}{32}\, q(z)f'(t)\left[4R^3 + 2R\,\frac{(R^2-R_1^2)}{\log\left(\frac{R_1}{R}\right)} - \frac{1}{R\log\left(\frac{R_1}{R}\right)}\left\{(4R^2R_1^2 - R_1^4 - 3R^4) - \frac{(R^2-R_1^2)}{R\log\left(\frac{R_1}{R}\right)}\left(4R_1^2\log\left(\frac{R_1}{R}\right) + 3R^2 - 3R_1^2\right)\right\}\right]$$

(25)

From equation (15), (22) and (23) the expression for volumetric flow rate is given by

$$Q(t) = \left[2(R^2 - R_1^2) + \frac{2R_1^2\log\left(\frac{R_1}{R}\right) - (R^2-R_1^2)}{\log\left(\frac{R_1}{R}\right)}\right]u_s + q(z)f(t)\left[(R^2-R_1^2)^2 - \frac{(R^2-R_1^2)}{\log\left(\frac{R_1}{R}\right)}\left\{\left(R_1^2 - 2R_1^2\log\left(\frac{R_1}{R}\right) - R^2\right)\right\}\right] + \frac{\alpha^2}{48}\, q(z)f'(t)\left[\{18R^4R_1^2 + 2R_1^6 - 12R^2R_1^4 - 8R^6\} - \frac{(R^2-R_1^2)}{\log\left(\frac{R_1}{R}\right)}\left\{6R^4 - 12R_1^4\log\left(\frac{R_1}{R}\right) + 12R_1^4 - 18R_1^2R^2\right\} + \left\{6R_1^2 + \frac{3(R^2-R_1^2)}{\log\left(\frac{R_1}{R}\right)}\right\} \times \left\{4R^2R_1^2 - R_1^4 - 3R^4 - \frac{(R^2-R_1^2)}{\log\left(\frac{R_1}{R}\right)}\left\{24R_1^2\log\left(\frac{R_1}{R}\right) + 18(R^2-R_1^2)\right\}\right\}\right]$$

(26)

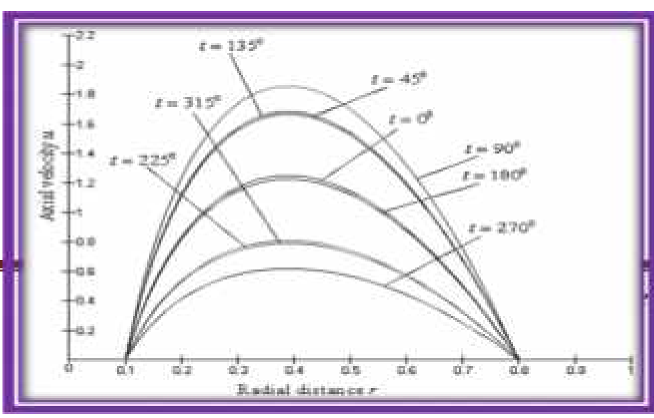
The effective viscosity $\boldsymbol{\mu_e}$ can be found out with the help of equations (17) and (26).

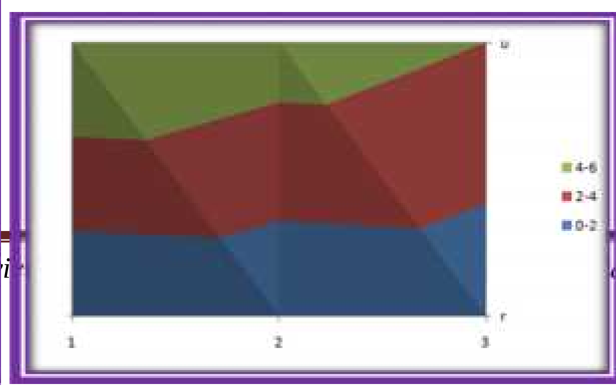
4. Results and Discussions

The present model has been investigated to analyse the effects of stenosis height, shape, catheter radius and slip velocity on axial velocity, shear stress and effective viscosity. The value 0.5 is taken for the amplitude *a* and the pulsatile Reynolds' number α, the range 0-0.2 is taken for the height of the stenosis δ_S. Radius of the catheter is taken in the range 0-0.5 and the value of the stenosis shape parameter is taken from 2 to 6.

Result: 1

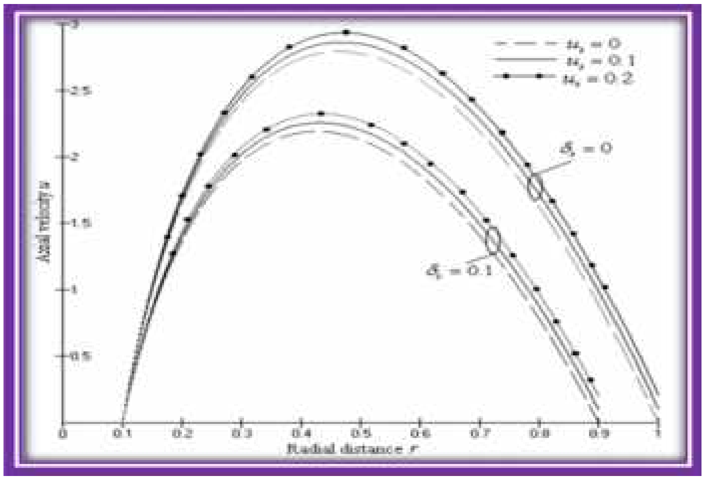
Fig. 2: Variation of axial velocity with radial distance for different values of *t*





It is found that velocity increases as the time t increases from $t = 0^0$ to $t = 90^0$ and then it decreases from $t = 90^0$ to $t = 270^0$ and then it again increases from $t = 270^0$ to $t = 360^0$.

Result: 2



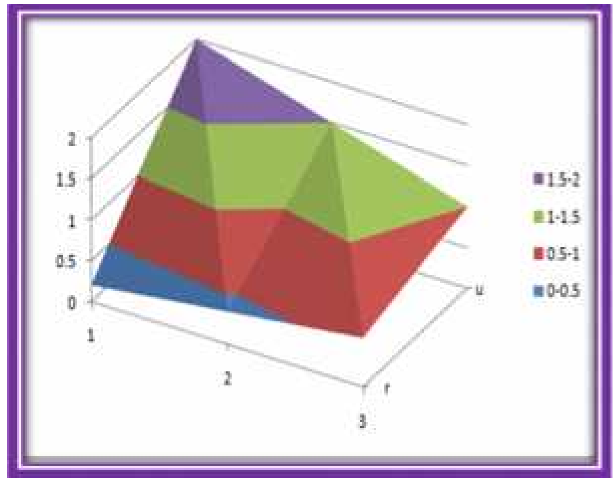
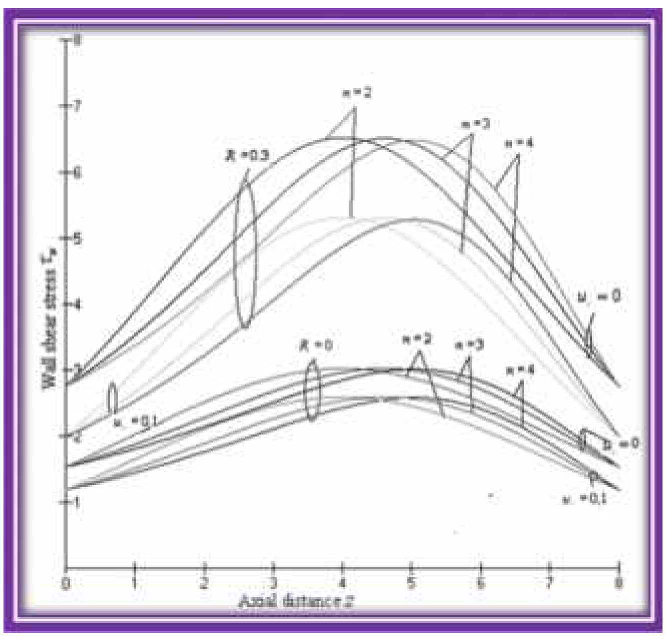


Fig.3: Variation of axial velocity with radial distance for different values of catheter radius R_1.

Velocity profiles for u in the case are found in catheterized artery than in an uncatheterized artery one for both uniform and stenosed arteries. Also, it is observed that magnitude of axial velocity significantly decreases with the increase in height of the stenosis in both catheterized and uncatheterized arteries.

Result: 3



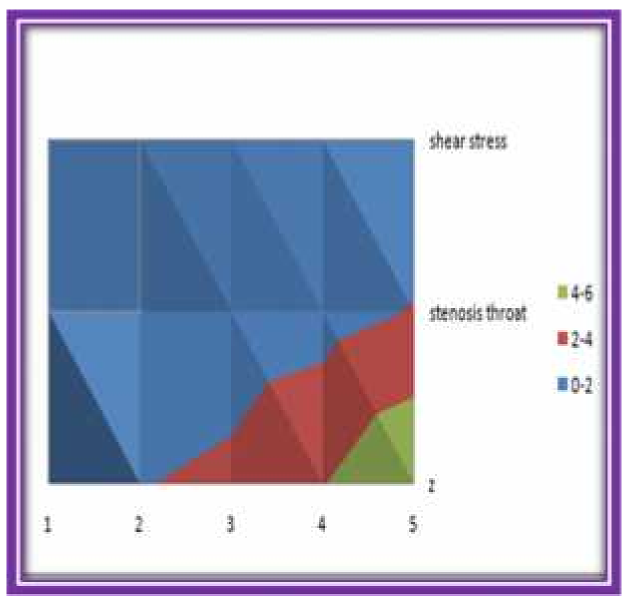
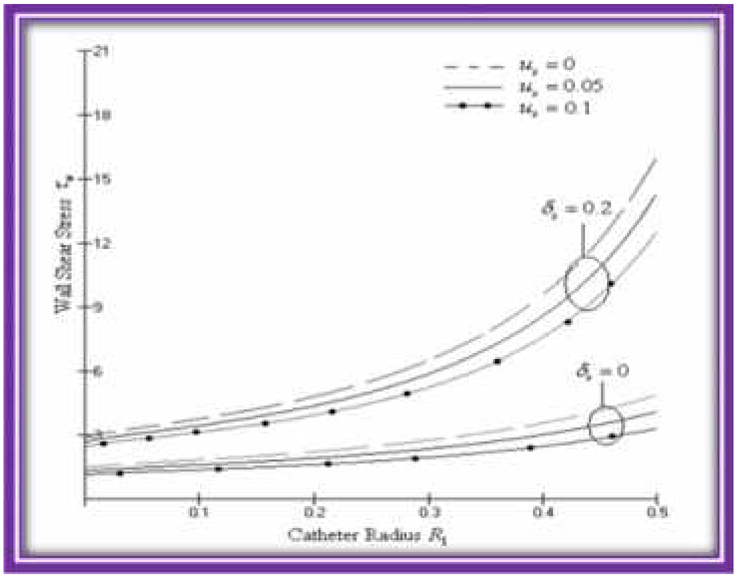


Fig. 4: Variation of wall shear stress with axial distance for different values of δ_s.

It is noted that the wall shear stress is very low in a uniform artery and it gradually increases wih the increase in stenosis size. It can be observed that τ_w increases as the axial distance z from -4 to 0 and then it decreases from 0 to 4. The maxixmum wall shear stress occurs at the throat of the stenosis (z = 0).

RESULT: 4

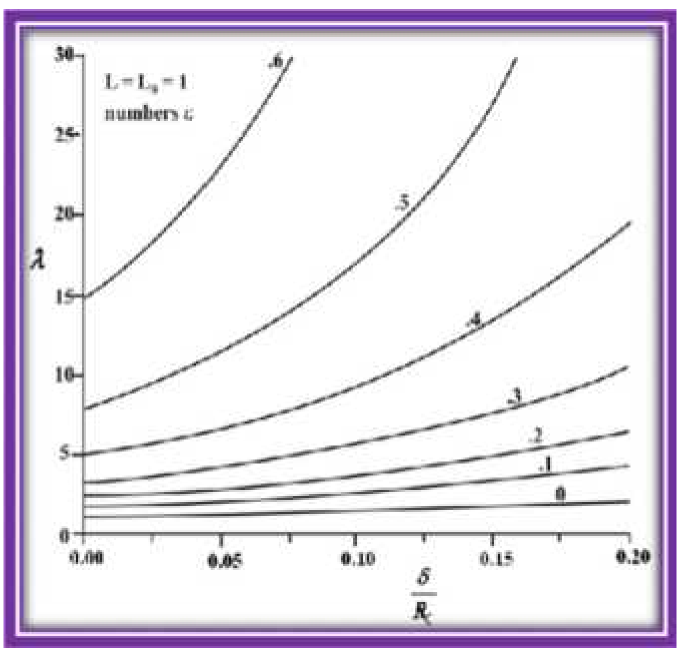


This shows the variation of effective viscosity with the radius of the inserted catheter for different values of shape parameter n and slip velocity u_s at $\delta_s = 0.2$, $t = 45^0$. It is found that effective viscosity increases with catheter radius significantly but decreases with shape parameter and slip velocity.

Fig. 5: Variation of wall shear stress with catheter radius for different values of δ_s.

RESULT: 5

Lo = L = 1(cm); (non- dimensional catheter radius = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, δ/ *R* (nondimensional stenosis height) = 0, 0.05, 0.10, 0.15, 0.20. (It is to note that the present study corresponds to the flow in un-catheterized and normal (no stenosis) artery for parameter values $\epsilon = 0$ and $\delta/R = 0$, respectively.



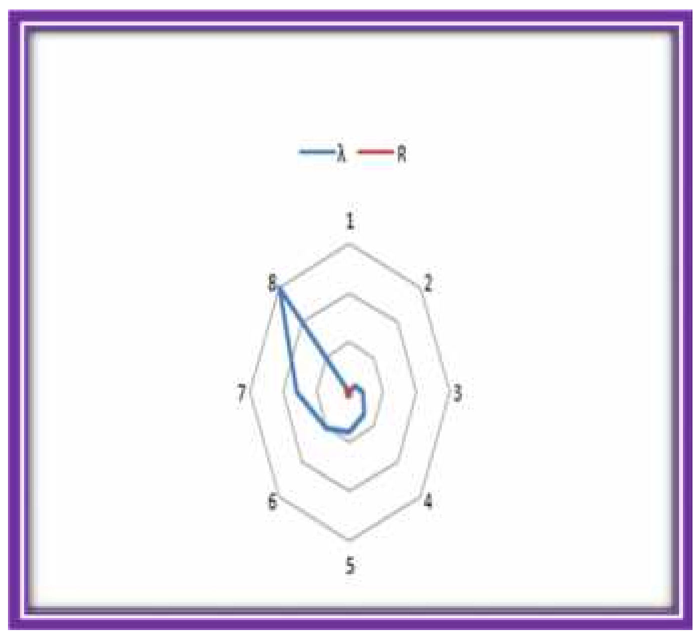
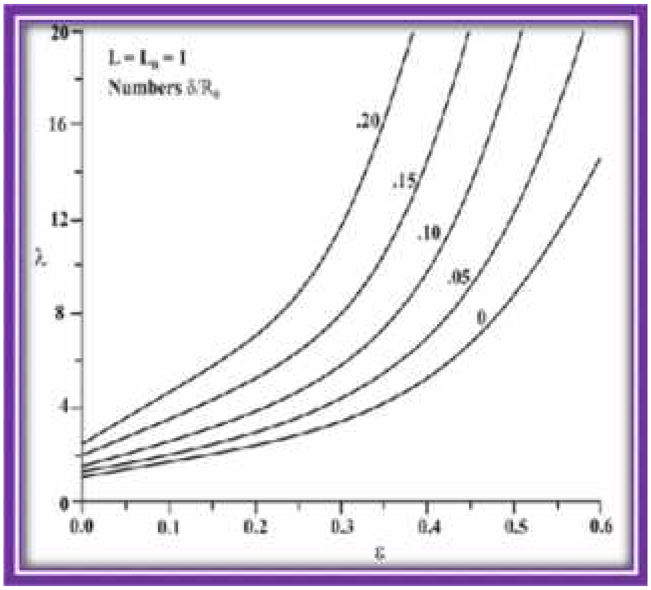


Fig. 6. Impedance, λ versus for different δ/R_0.



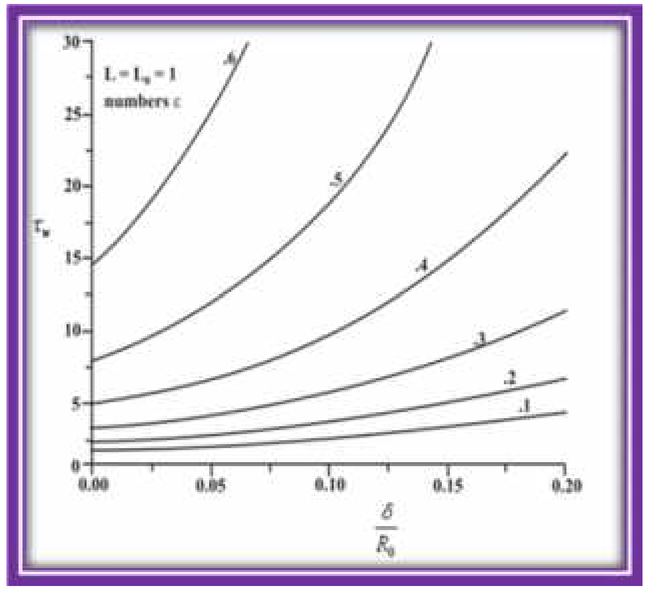


Fig. 7. Impedance, λ versus for different δ/R_0.

Fig. 8. Shear stress at stenosis throat τ_w versus δ/R_0.

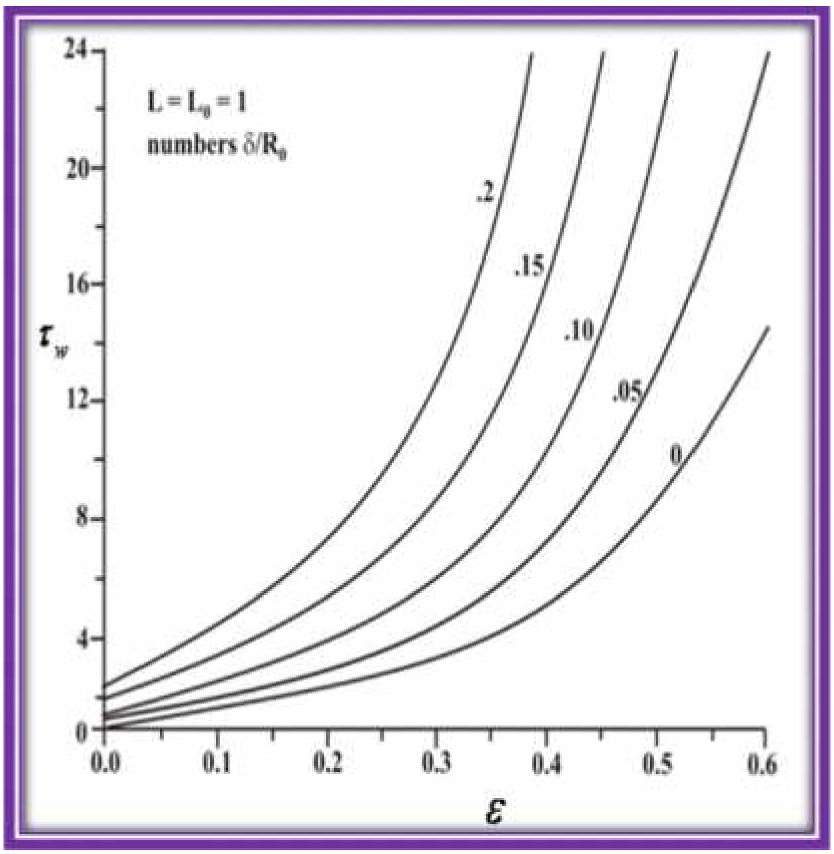


Fig. 9. Shear stress at stenosis throat τ_w versus ϵ.

The high asymptotic magnitude of λ occurs for δ/R = 0.1(19% *stenosis*), 0.15 (28% stenosis) and 0.2 (38% stenosis) at catheter size, ϵ = 5.5, 4.5 *and* 4.0 respectively. Also notice that τ_w is increases with increasing stenosis height,δ/R_0 (Fig.8). The wall shear stress in the stenotic region, τ_w increases rapidly with the catheter size,ϵ.(Fig. 9).

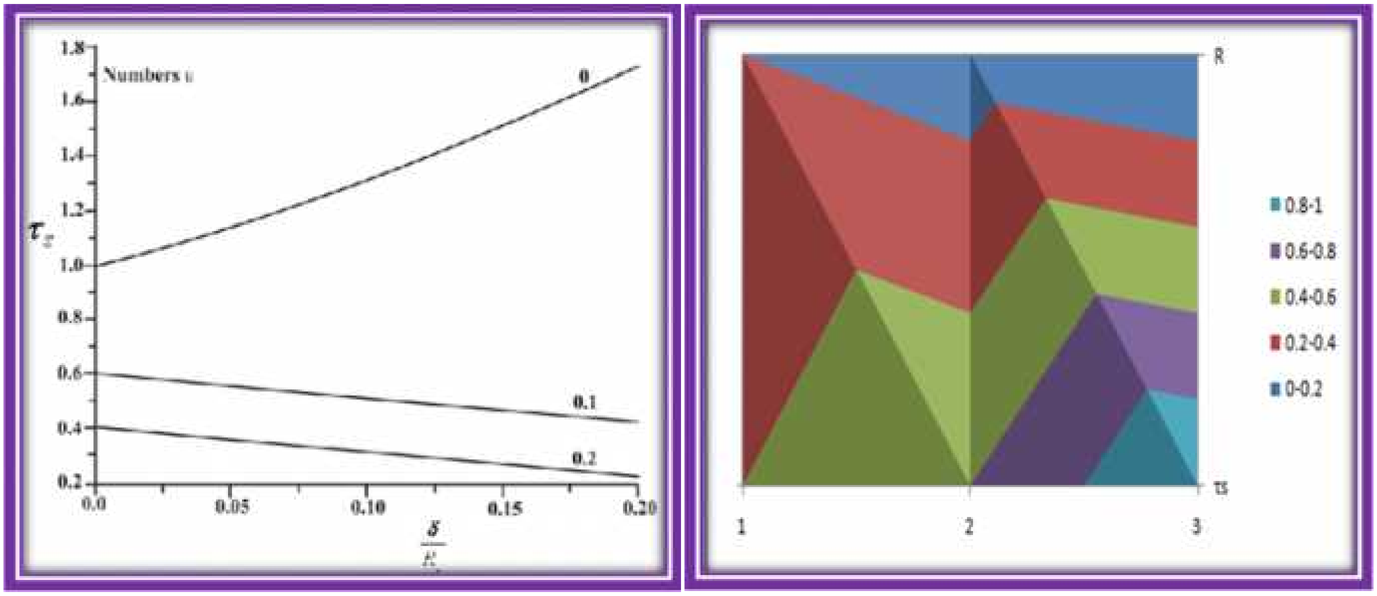
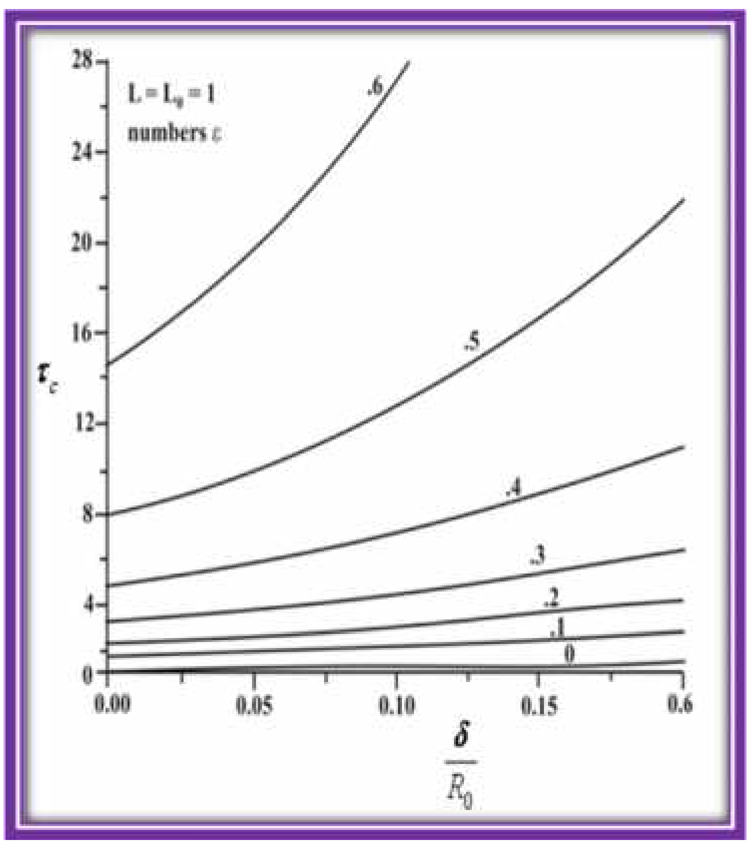
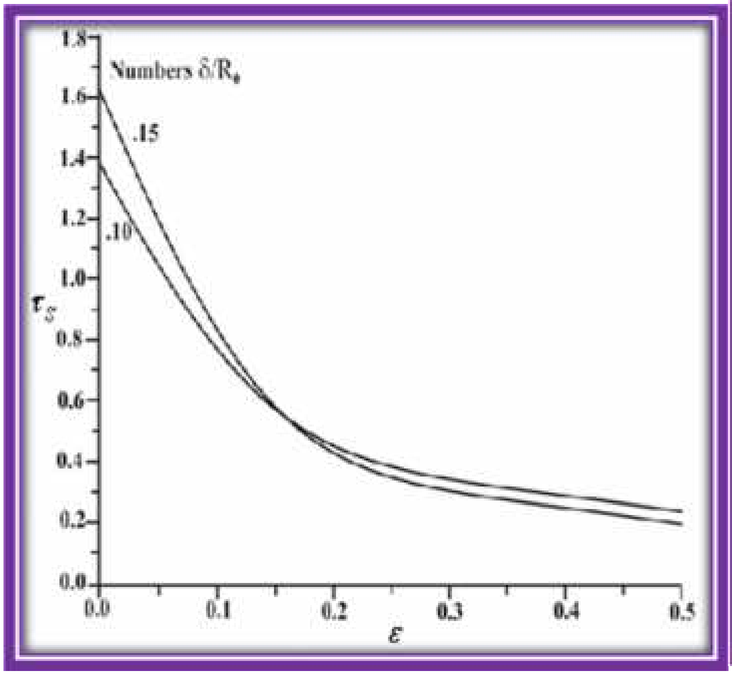


Fig. 10. Variation of shear stress at stenosis throat τ_w with δ/R_0 for different values of ϵ.



The shear stress at the stenosis throat τ_s increases with the catheter size, ϵ as well as with the stenosis height, δ/R (Fig. 10). At stenosis critical height, the shear stress τ_c too increases with the catheter size, and stenosis height δ/R (Fig. 11). The flow characteristic, τ_s assumes higher magnitude for higher stenosis height for small catheter size, ϵ (Fig. 12). One notices that τ_s achieves an asymptotic magnitude when the catheter size becomes approximately fifty percent of the artery size.

Fig. 11. Shear stress at stenosis critical height, τ_s versus δ/R_0 for different values of ϵ.



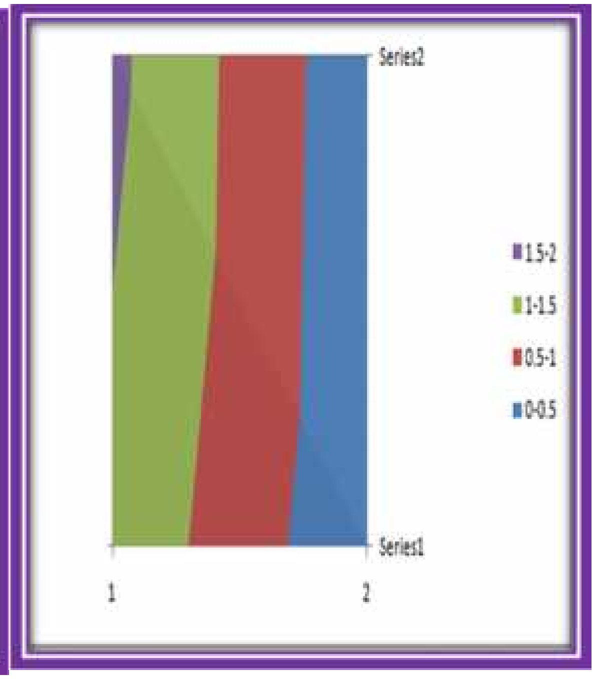


Fig. 12. Variation of shear stress at stenosis throat τ_w for different values of δ/R_0 .

Conclusion

In the present work, the increased impedance and shear stress during a narrow catheterized has been analyzed assuming that the flowing blood is considered as a Newtonian fluid. Graphical results shows that wall shear stress and effective viscosity decrease while axial velocity increases with velocity slip at wall; magnitude of axial velocity significantly decreases with the increase in height of the stenosis in both catheterized and uncatheterized arteries; the wall shear stress is very low in a uniform artery and it gradually increases with the increase in stenosis size; the wall shear stress goes on increasing with the increase in stenosis size and the radius of the catheter. Thus, the magnitude of the stenosis critical height is smaller than its corresponding value at the stenosis throats.

References

1. D. Srinivasacharya and D. Srikanth, Effect of couple stresses on the flow in a constricted annulus, Arch Appl Mech., 78 (2008), 251–257.
2. D.A. McDonald, Pulsatile flow in a catheterized artery, J. Biomech., 19,(1986), 239-249.
3. D. F. Young, Effects of a Time-dependent Stenosis of Flow through a Tube, Journal of Eng. Ind., 90 (1968), 248-254.
4. D. S. Sankar, and K. Hemalatha, A Non-Newtonian fluid flow model for blood flow through a catheterized artery - Steady flow, Applied Mathematical Modeling, 31 (2007), 1847 – 1864.
5. D.S. Sankar, A two-fluid model for pulsatile flow in catheterized blood vessels, International Journal of Non-Linear Mechanics, 44 (2009), 337 – 351.
6. E.W. Merrill, Rheology of Human Blood and Some Speculations on its Role in Vascular Homeostasis Biomechanical Mechanisms in Vascular Homeostasis and Intravascular Thrombosis. P.N. Sawyer (ed.). Appleton Century Crofts, New York, (1965), 127-137.
7. D.S. Sankar and U. Lee, Mathematical modeling of pulsatile flow of non-Newtonian fluid in stenosed arteries, Commun Nonlinear Sci Numer Simulat., 14 (2009), 2971-2981.

8. G. Jayaraman and R.K. Dash, Numerical study of flow in a constricted curved annulus: An application to flow in a catheterised artery, J. Eng. Math.,40 (2001), 355-376.
9. G.T. Karahalios, Some possible effects of a catheter on the arterial wall, Med. Phys., 17 (1990), 925-992.
10. M.G. Taylor, The Influence of Anomalous Viscosity of Blood upon its Oscillatory Flow, Physics in Medicine and Biology, 3(1959), 273-290.
11. R.K. Dash and G. Jayaraman and K.N. Mehta, Estimation of increased flow resistance in a narrow catheterized artery - a theoretical model, J. Biomech.,29(1966), 917-930.
12. S. Chakraborty, A. Datta and P.K. Mandal, Analysis of Non-linear Blood flow in a Flexible artery, Int. J. Eng. Sci., 33 (1995),1821-1837.
13. V.P. Srivastava and R. Rastogi, Effects of Hematocrit on Impedance and Shear Stress during Stenosed Artery Catheterization, Appl. Appl. Math., 4,(2009), 98-113.
14. V.K. Sud and G. S. Sekhon, Arterial flow under periodic body acceleration, Bulletin of Mathematical Biology, 47(1985), 35-52.
15. P. Nagarani, and G. Sarojamma, Effect of body acceleration on pulsatile flow of Casson fluid through a mild stenosed artery, Korea-Australia Rheology Journal, **20** (2008), 189-196.
16. D. Biswas, Blood Flow Models: A Comparative Study, Mittal Publications,New Delhi, 2000.
17. D.F. Young, Fluid Mechanics of Arterial Stenosis, J. Eng. Ind. Trans. ASME,**101**(1979), 157- 175.
18. D. Biswas and U.S. Chakraborty, Pulsatile Flow of Blood in a Constricted Artery with Body Acceleration, Appl. Appl. Math., **4**(2009), 329-342.