PLANARITY IN GRAPHS AND GRAPH COLORING

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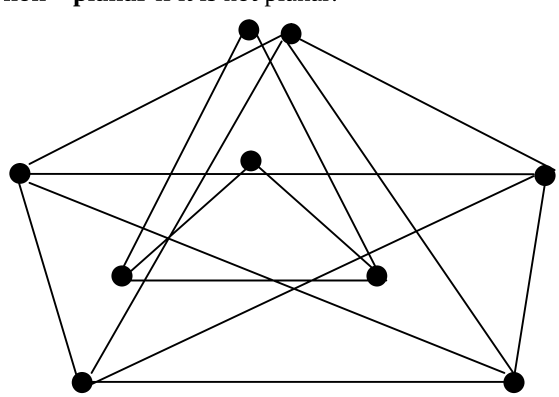
Abstract
Planar graphs comprise a quite uncomplicated category of graphs and planarity is one of the innermost concept of the entire graph theory, so just entirely from the theoretical point of view it is interesting to consider planar graphs in algorithmic framework. A graph is planar if it can be embedded in the plane so that no two edges intersect geometrically except at a vertex to which they are both incident. In this study we survey the results on planar graphs.

Keywords: Planar, Embedded.

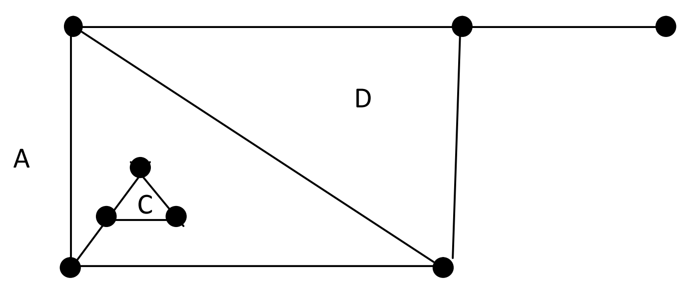
Introduction
The concept of planar graphs occurs in real world applications, among others to represent images, construction plans and traffic maps. Planar graphs satisfy interesting properties which allow much more efficient processing than general graphs. Planar graphs relate to some of the most exciting parts of graph theory. A graph is planar if it can be drawn in the plane without any crossing edges. That is, each vertex is located at one point of the plane, and a curve from one point to another is drawn between the points corresponding to vertices connected by an edge. None of these curves intersect each other, and may only touch at the points representing the vertices at which they start or end. A more practical reason for studying planar graphs is that they, and their relatives, appear in many practical applications. The study of two-dimensional images often results in problems related to planar graphs, as does the solution of many problems on the two-dimensional surface of our earth. Many natural three-dimensional graphs arise in scientific and engineering problems.

Preliminaries
Definition 1.1: A graph is said to be **embedded** in a surface S when it is drawn on S so that no two edges intersect.
Definition 1.2: A graph is called **planar** if it can be drawn on a plane without intersecting edges.

Example
Definition 1.3: A graph is called **non – planar** if it is not planar.
Example



Definition 1.4: The **face** of a planar graph is the areas which are surrounded by edges.
Example :



Here A,B,C,D are the faces

Definition 1.5: A planar graph is called **outer planar** if it can be embedded in the plane so the all its vertices lie on the same face. This face is often chosen to be the exterior face.

Definition 1.6: An outer planar graph is called **maximal outer planar** if no line can be added without losing outer planarity. Every maximal outer planar graph is a triangulation of a polygon while every maximal plane graph is triangulation of the sphere.

Definition 1.7: A graph is **spherical** if it can be embedded on the surface of a sphere.

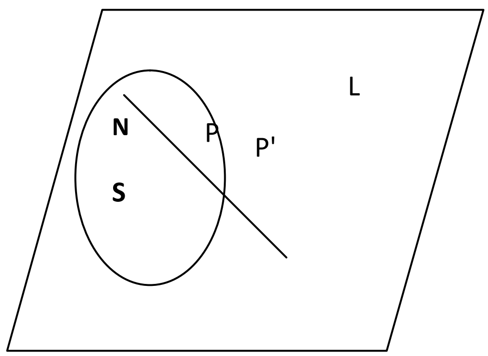
Planar and Non-Planar Graphs

Theorem 1: A graph can be embedded in the surface of a sphere if and only if it can be embedded in a plane.

Proof: Let G be a graph can be embedded on a sphere.

Place the sphere on a plane L and call the point of contact S (South Pole).

At points, draw a normal to the plane and let N (North Pole) be the point where this normal intersects the surface of the sphere.

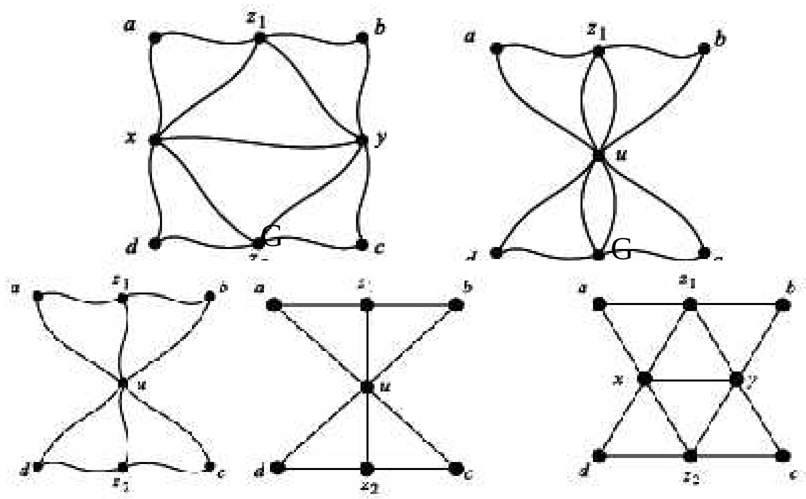


Assume that the sphere is placed in such a way that N is disjoint from G.
For each point P on the sphere, let P^I be the unique point on the plane, where the line NP intersects the surface of the plane. Thus there is a one to one correspondence between the points of the sphere other than N and the points on the sphere. (P^I is called the stereographic projection of P on L). In this way the vertices and the edges of G can be projected on the plane L, which gives an embedding of G in L. The reverse process obviously gives an embedding in the sphere for any graph that is embedded in the plane L.

Theorem 2: Every triangulated planar graph has a straight line representation.[2].

Proof

The proof is by induction on the number of vertices. The result is obvious for $n = 4$. So, let $n \geq 5$ and assume that the result is true for all planar graphs with fewer than n vertices. Let G be a plane graph with n vertices. First, we show that G has an edge e belonging to just two triangles. For this, let x be any vertex in the interior of a triangle T and choose x and T such that the number of regions inside T is minimal. Let y be a neighbor of x, and the edge xy lies inside T, and let xy belong to three triangles xyz_1, xyz_2 and xyz_3. Then one of these triangles lies completely inside another. Assume that z_3 lies inside xyz_1. Then z_3 and xyz_1 contradict the choice of x and T.



G H'' H

Thus there is an edge $e = xy$ lying in just two triangles xyz_1 and xyz_2. Contracting xy to a vertex u, we get a new graph G' with a pair of double edges between u and z_1, and u and z_2. Remove one each of this pair of double edges to get a graph G'' which is a triangulated graph with $n-1$ vertices.

By the induction hypothesis, it has a straight line representation H''. The edges of G'' correspond to uz_1, uz_2 in H''. Divide the angle around u into two parts in one of which the pre-images of the edges adjacent to x in G lie, and in the other, the pre-images of the edges adjacent to y in G.

Thus u can be pulled apart to x and y, and the edge xy is restored by a straight line to get a straight line representation of G.

Theorem 3: The complete graph K_5 with five vertices is non-planar.

Proof: Let the five vertices in the complete graph be named v_1, v_2, v_3, v_4, v_5. Since in a complete graph every vertex is joined to every other vertex by means of an edge, there is a cycle $v_1\ v_2\ v_3\ v_4\ v_5\ v_1$ that is a pentagon. This pentagon divides the plane of the paper into two regions, one inside and the other outside, Figure a.

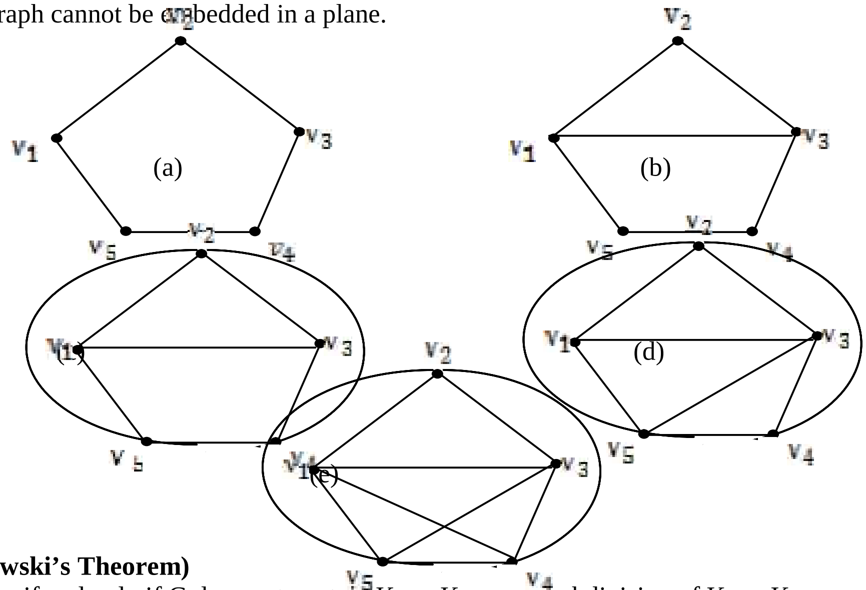
Since vertex v_1 is to be connected to v_3 by means of an edge, this edge may be drawn inside or outside the pentagon (without intersecting the five edges drawn previously). Sup-pose we choose to draw the line from v_1 to v_3 inside the pentagon, Figure (b).

In case we choose outside, we end with the same argument. Now we have to draw an edge from v_2 to v_5 and another from v_2 to v_5.

Since neither of these edges can be drawn inside the pentagon without crossing over the edge already drawn, we draw both these edges outside the pen-tagon, Figure (c).

The edge connecting v_3 and v_5 cannot be drawn outside the pentagon without crossing the edge between v_2 and v_4. Therefore v_3 and v_5 have to be connected with an edge inside the pentagon, Figure (d).

Now, we have to draw an edge between v_1 and v_4 and this cannot be placed inside or outside the pentagon without a crossover. Thus the graph cannot be embedded in a plane.

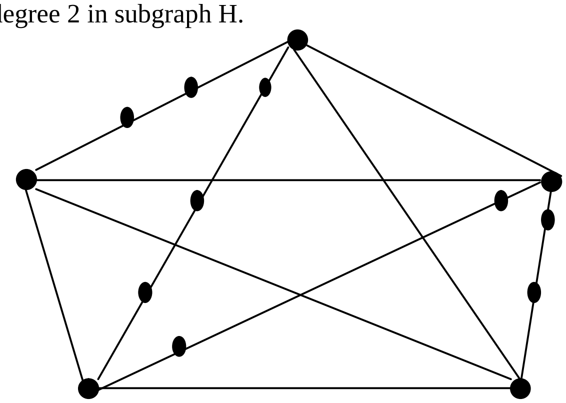


Theorem 4: (Kuratowski's Theorem)
[1] A graph G is planar if and only if G does not contain K_5 or $K_{3,3}$, or a subdivision of K_5 or $K_{3,3}$ as a subgraph.

Proof: Suppose that we are given a graph G of order n ≥ 3 and size m and we wish to determine whether G is planar. To show that G is planar.

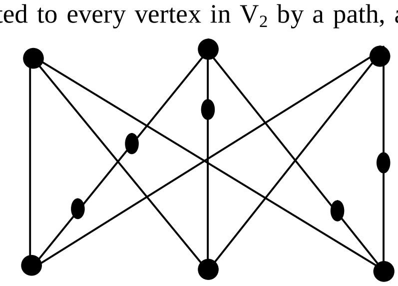
Certainly one point is to draw G as a plane graph. One way to verify that G is non planar is to show that $m > 3n - 6$. However if $m \leq 3n - 6$ it may still be the case that G is non planar.A certain way to verify that G is non planar is to show that K_5 or $K_{3,3}$ is a subgraph of G or a subdivision of K_5 or $K_{3,3}$ as a subgraph of G. To show that G contains a subdivision of K_5 as a subgraph.We need to find a subgraph H containing five vertices of degree 4, every two of which are connected by a path all of whose interior vertices have degree 2 in subgraph H.

(a)



To show that G contains a subdivision of $K_{3,3}$ as a subgraph.
We need to find a subgraph F containing six vertices of degree 3, partitioned into two sets V_1 and V_2 of three vertices each, such that every vertex in V_1 is connected to every vertex in V_2 by a path, all of whose interior vertices have degree 2 in a subgraph F.

(b)

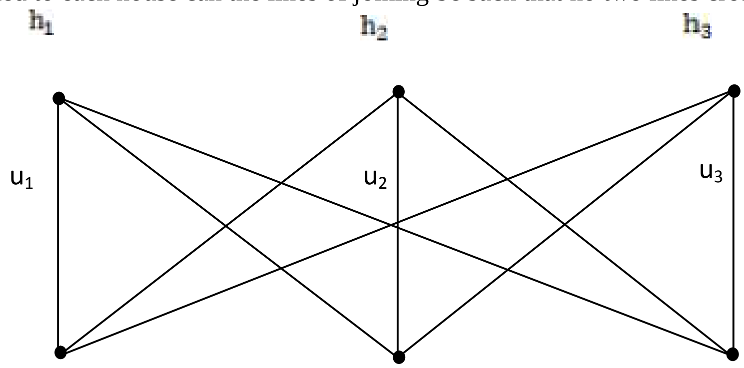


This also says that if G is a graph that contains

a) at most four vertices of degree 4 or more and
b) at most five vertices of degree 3 or more Then G must be planar

Example 1: Suppose there are three houses and three utility points (electricity, water, gas say) which are such that each utility point is joined to each house can the lines of joining be such that no two lines cross each other.

Soln:

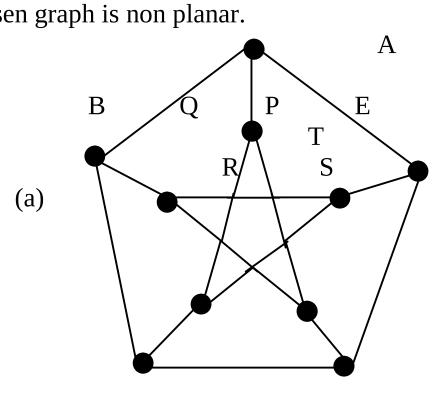


Consider the graph in which the vertices are three houses (h_1, h_2, h_3) and three utility points (u_1, u_2, u_3) .Since each house is joined to each utility point, the graph has to be $K_{3,3}$

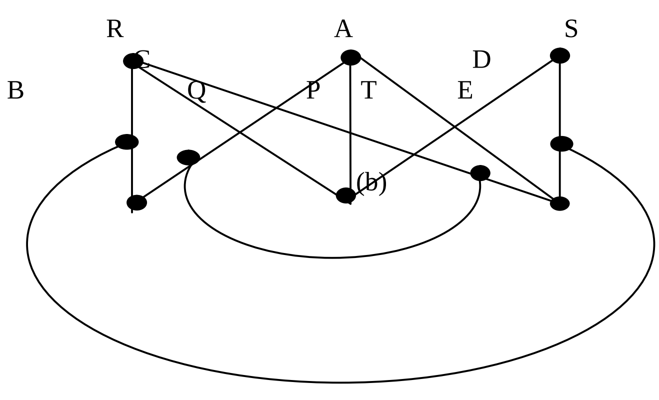
This graph is non planar. Therefore in its every plane drawing at least two of its edges cross each other. It is not possible to have the lines joining the houses and the utility points such that no two lines cross each other.

Example 2: Petersen graph is non planar.

Proof



Petersen graph is a 3-regular graph of order 10 and size 15.
In this graph vertices are named as A, B, C, D, E.



Now consider the graph (b)
We verify that this graph is another representation of the Peterson graph.
The resulting graph is $K_{3,\ 3.Thus}$ the Petersen graph contains $K_{3,3}$ as a subgraph.
Therefore the Peterson graph is non planar.

Characterization of Planar Graphs
Theorem 5: (Euler's Formula) If G = (V, E) is a connected planar graph with n (≥ 1) vertices, m edges, and r regions, then n−m+r=2.

Proof
Basis
Consider the case m=0.
In this case, n must be 1 since G is connected.
With 1 vetex and no edges, there is exactly one outer region.
Thus n−m+r=1−0+1=2 as desired.

Inductive step:
Assume that the statement is true for any connected planar graph with at most k edges.
In other words, for any connected planar graph G = (V, E) with n vertices m ≤ k, edges, and r regions, n−m+r=2. This is the inductive hypothesis.
Suppose G = (V, E) is a connected planar graph with n vertices, m = k + 1 vertices, and r regions.If G is a tree, then m = n − 1.
Notice that if G is the tree, then the only region is the outer region surround the tree so r = 1.
Thus n− m + r = n− (n−1) + 1 = 2 as desired.
If G is not a tree, then G has some cycle. Call the cycle C.
Let e be any edge in C, and consider a subgraph G' = (V, E\{e}) of G.
G' must be connected because removing an edge from a cycle breaks the cycle but does not break the connectivity of the graph. Furthermore, G' has one fewer region than G because removing an edge merges two regions into one region.
Because subgraph of a planar graph is also planar and |E| = m = k + 1,
G' is a connected planar graph with |E\{e}| = k edges.
Thus the inductive hypothesis applies to G' to yield n − k + (r−1) = 2.
Therefore, n−m+r = n − (k + 1) + r = n – k + (r − 1) = 2.
Thus in either case, n – m + r = 2 as required.

Therorem 6: If G is a simple planar graph, then G has a vertex v of degree less than 6.[5].

Proof
If G has only one vertex, then this vertex has degree zero. If G has only two vertices, then both vertices have degree at most one.

Let $n \geq 3$. Assume degree of every vertex in G is at least six.

Then, $\sum_{v \in V(G)} d(v) \geq 6n$.

We know $\sum_{v \in V(G)} d(v) \geq 2m$.

Thus, $2m \geq 6n$ so that $m \geq 3n$. we have $m \leq 3n - 6$. Thus we get a contradiction. Hence G has at least one vertex of degree less than 6.

Example 3: What is the minimum number of vertices necessary for a simple connected graph with 11 edges to be planar?
Solution:
For a simple connected planar graph G(n, m)
We have m ≤ 3n-6
3n ≥ m + 6
n ≥ m + 6/3

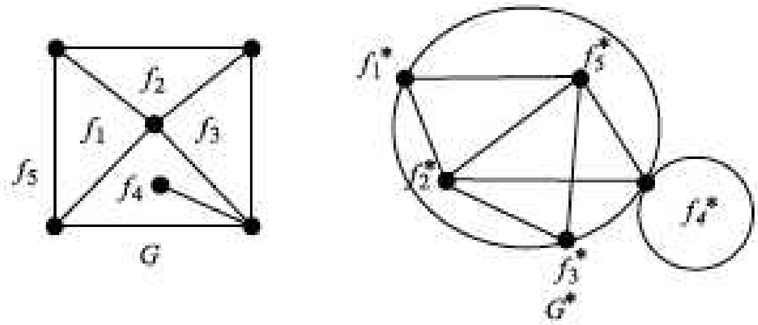
When m = 11 we get, $n \geq 1/3\ (11+6)$

$n \geq 17/3$

Hence the required minimum number of vertices 6.

Duals in Planar Graphs

Definition: Let G be a plane graph. The dual of G is defined to be the graph G^* constructed as follows. To each region f of G there is a corresponding vertex f^* of G^* and to each edge e of G there is corresponding edge e^* in G^* such that if the edge e occurs on the boundary of the two regions f and g, then the edge e^* joins the corresponding vertices f^* and g^* in G^*. If the edge e is a bridge, i.e., the edge e lies entirely in one region f, then the corresponding edge e^* is a loop incident with the vertex f^* in G^*. For example, consider the graph shown in Figure.



Theorem 7: The dual G^* of a plane graph is planar.

Proof

[3] Let G be a plane graph and let G* be the dual of G. The following construction of G* is planar.

Place each vertex f_k^* of G^* inside its corresponding region f_i. If the edge e_i lies on the boundary of two regions f_j and f_k of G, join the two vertices f_j^* and f_k^* by the edge e_i^*, drawing so that it crosses the edge e exactly once and crosses no other edge of G

Remarks Clearly, there is one-one correspondence between the edges of plane graph G and its dual G^* with one edge of G^* intersecting one edge of G.

1.An edge forming a self-loop in G gives a pendant edge in G^* (An edge incident on a pendant vertex is called a pendant edge).

A pendant edge in G gives a self loop in G^*.
Edges that are in series in G produce parallel edges in G^*.
Parallel edges in G produce edges in series in G^*.
The number of edges forming the boundary of a region f_i in G is equal to the degree of the corresponding vertex f_i^* in G^*.Considering the process of drawing a dual G^* from G, it is evident that G is a dual of G. Therefore, instead of calling G^* a dual of G, we usually say that G and G* are dual graphs. Let n,m,f,r and μ denote the number of vertices, edges, regions, rank and nullity of connected plane graph G and let n^*,m^*,f^*,r^* and μ^* be the corresponding numbers in G*. Then $n^* = f$, $m^*=m$, $f^*=n$. Using Euler's formula, n-m+f=2 or f=m-n+2, We have $r^*=m-n-2-1=m-n+1=\mu$ and $\mu^*=m-f+1=n+f-2-f+1=n-1=r$.

Theorem 8: The edge e is a loop in G if and only if e^* is a bridge in G^*.

Proof

Let the edge e be a loop in a plane graph G. Then it is the edge on the common boundary of two regions on which, say f , lies within the area of the plane surrounded by e with the other, say g, lying outside the area. Thus, from definition of G^*, e^* is the only path from f^* to g^* in G^*. Thus e^* is a bridge in G^*. Conversely, let e^* be a bridge in G^*, joining vertices f^* and g^*.
Thus e^* is the only path in G^* from f^* to g^*.

This implies, again from the definition of G^*, that the edge e in G completely encloses one of the regions f and g. So e is a loop in G.

Coloring in Planar Graphs

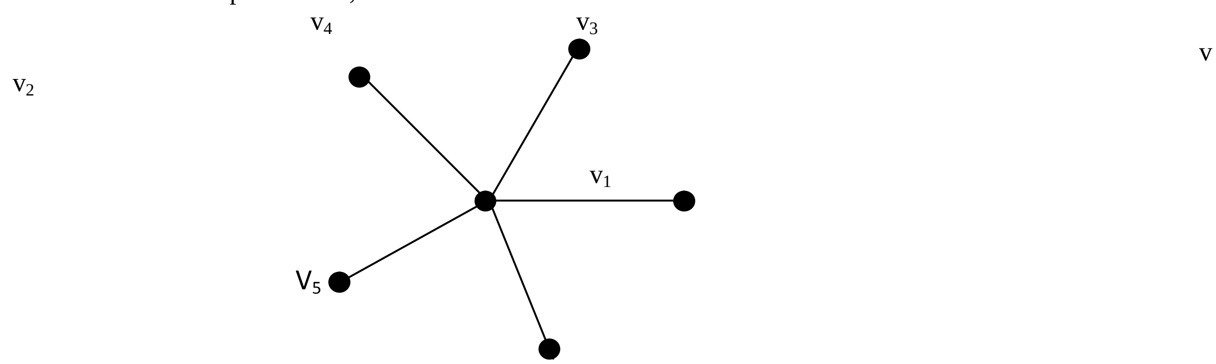
Definition: A **coloring** of a graph G assigns a color to each vertex of G, with the restriction that two adjacent vertices never have the same color.

Definition: The chromatic number of G, written χ(G), is the smallest number of colours needed to colour G.

Theorem 9: (Five colors Theorem): The vertices of every connected simple plane graph can be properly coloured with five colors.

Proof:

Let n be the number of vertices in a connected simple planar graph.
If $n \leq 5$, then the theorem is trivially true.
Assume that the theorem is true for all graphs with $n \leq k$.
Consider a graph G with k + 1 vertices.
Then by Euler's theorem, G contains a vertex v of degree at most 5.
If we consider the graph H = G – v, obtained by deleting v from G, then H has k vertices.
Therefore the assumption made, H is 5 – colourable.



Since the degree of v is at most 5, v has at most 5 neighbors in G.
Suppose v has 4 or less number of neighbours.
Then the neighbours can be coloured with at most four different colours and v can be coloured with fifth colours, all drawn from the colours used in H.

Thus a proper colouring of G can be done by using the five colours with which H can be coloured.Thus G is 5 – colourable.
Next suppose that v has 5 neighbours, say v_1, v_2, v_3, v_4, v_5.Let us arrange them around v in anti-clockwise order.If the vertices v_1, v_2, v_3, v_4, v_5 are all mutually adjacent, then they constitute K_5 which is non planar.This is not possible, because being a planar graph, G cannot contain a planar graph as a subgraph. Therefore, at least two of v_1, v_2, v_3, v_4, v_5 say v_1 and v_3 are non adjacent.Now construct a graph G' by merging the edges v_3v and vv_1.

The graph G' will have (k + 1) – 2 = k – 1vertices (with v_3vv_1 as the merged vertex).
Therefore this graph is 5 – colourable.Let us assign a colour α_1 to the merged vertex v_3vv_1, a colour α_2 to v_2, a colour α_4 to v_4 and a colour α_5 to v_5.With this scheme of colouring of v_1, v_2, v_3, v_4, v_5 and with the use of just one more colour α_3 assigned to other appropriate vertices, the graph G' gets properly coloured.Now, unravel the merged vertex v_3vv_1 and assign the colour α_1 to both v_3 and v_1 and the colour α_3 to v, without disturbing the colours of other vertices.This will produce a proper colouring of G with colours α_1, α_2, α_3, α_4, α_5.

Thus, G is 5-colourable in this case also (where the degree of v is 5).
We have proved that a graph with n = k + 1 vertices is 5-colourable if a graph with $n \leq k$ vertices is 5-colourable.Hence by induction, it follows that a graph with n vertices, where n is any positive integer, is 5-colourable.This completes the proof of the theorem.

Conclusion

This study describes the planar, non-planar graphs, dual graph and very important characteristics of a planar graphs. Using the concept of a planar graph we can define dual planar graph in such a way that if the edge e occurs on the boundary of the two regions f and g, then the edge e^* joins the corresponding vertices f^* and g^* in G^*.Also graph coloring is an interesting concept in planar graph. In this article coloring and the detailed concept of dual is exemplified.

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